

Quasi-algorithmical construction of reciprocal transformations

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Abstract

Reciprocal transformations mix the role of the dependent and independent variables to achieve simpler versions or even linearized versions of nonlinear PDEs. These transformations help in the identification of a plethora of PDEs available in the Physics and Mathematics literature. Two different equations, although seemingly unrelated, happen to be equivalent versions of a same equation after a reciprocal transformation. In this way, the big number of integrable equations could be greatly diminished by establishing a method to discern which equations are disguised versions of a common underlying problem. Then, a question arises: Is there a way to identify different versions of an underlying common nonlinear problem? Other useful applications of reciprocal transformations are subsequently discussed and illustrated with examples.

1 Introduction

Reciprocal transformations are a type of transformation that exchange the role of dependent and independent variables [8, 19]. When the role of dependent and independent variables is switched, the final space of independent variables is called a *reciprocal space*. As a physical interpretation, whereas the independent variables play the role of positions in the reciprocal space, this number is increased by turning certain dependent variables into independent variables and viceversa [9]. In particular, in the case of reciprocal transformations, the existence of conserved quantities for their quasi-algorithmical proposal is fundamental [12, 13, 20, 21, 30, 31, 32]. These conservation laws are present in models of real physical processes, where reciprocal transformations are applicable.

During the past decades, more attention has been laid upon reciprocal transformations, due to their manageability, quasi-algorithmical way of

approach [12, 20, 21]. Indeed, the interest in the topic is reflected in a growing number of works [1, 3, 12, 20, 21, 22, 23, 24, 26]. For example, reciprocal transformations were proven to be a useful instrument to transform equations with peakon solutions into equations that are Painlevé integrable [10, 26]. In 1928, the invariance of nonlinear gas dynamics, magnetogas dynamics and general hydrodynamic systems under reciprocal transformations was extensively studied [22, 33]. Stationary and moving boundary problems in soil mechanics and nonlinear heat conduction have likewise been subjects of research [24, 29]. Applications of the reciprocal transformation in continuum mechanics are to be found in the monographs by Rogers and Shadwick [35]. These transformations have also played an important role in the soliton theory and between hierarchies of PDEs [10, 26]. Furthermore, the invariance of certain integrable hierarchies under reciprocal transformations induces auto-Bäcklund transformations [20, 21, 27, 32, 34]. Some of the most representative properties of reciprocal transformations are they map conservation laws to conservation laws and diagonalizable systems to diagonalizable systems, but act nontrivially on metrics and on Hamiltonian structures. For instance, the flatness property or the locality of the Hamiltonian structure are not preserved, in general [2, 3].

Finding a proper reciprocal transformation is usually a very complicated task. Notwithstanding, in the cases that concern us: as it can be the case of equations in fluid mechanics, a change of this type is usually reliable. For systems of hydrodynamic type with time evolution

$$(u_j)_t = \sum_{l=1}^k v_l^j(u)(u_l)_{x_i}, \quad \forall 1 \leq i \leq n, \quad j = 1, \dots, k \quad (1)$$

and $v_l^j(u)$ are infinitely differentiable functions. These evolution equations appear in gas dynamics, hydrodynamics, chemical kinetics, differential geometry and topological field theory [11, 37].

Another advantage of dealing with reciprocal transformations is that many of the differential equations reported integrable in the bibliography of differential equations, which are considered seemingly different from one another, happen to be related via reciprocal transformations. If this were the case, two apparently unrelated equations, even complete hierarchies of PDEs, can be tantamount versions of a unique problem. In this way, reciprocal transformations give rise to a procedure of relating allegedly new equations to the rest of their equivalent integrable sisters. It is an attempt to diminish the “zoo-botanical approach” in which a number of integrable, specially nonlinear PDEs have been cultivated by slight modifications in pa-

rameters that still fit the integrability conditions of algorithmic, algebraic or geometric nature. One suspected way to classify equivalent equations is to find a canonical form shared by all those equations that are related via a reciprocal transformation. It is intuitively based on the singular manifold method (SMM) [14, 18] and some clues have already been depicted in [36].

A second significant application of reciprocal transformations is their utility in the identification of differential equations which are not integrable according to algebraic tests (for example, the Painlevé test is one of them) [13, 19]. Precisely, our motivation for the study of reciprocal transformations rooted in the study of the Camassa-Holm hierarchy in $2+1$ dimensions [16]. This hierarchy, has been known to be integrable for some time and has an associated linear problem. Nevertheless, in its most commonly expressed form [16] it is not integrable according to the Painlevé test, nor the SMM is constructive. Our conjecture is that if an equation is integrable, there must be a transformation that will let us turn the initial equation into a new one in which the Painlevé test is successful and the SMM could be applied. From here, a Lax pair could be derived, among many other properties [36].

2 General Setting

Let us consider a general manifold $N_{\mathbb{R}^n} \simeq \mathbb{R}^k \times \mathbb{R}^n$, where the first k -tuple refers to the dependent variables $u = (u_1, \dots, u_k) \in \mathbb{R}^k$ and the n -tuple denotes the independent variables $x = (x_1, \dots, x_n) \in \mathbb{R}^n$. We denote by $J^p(\mathbb{R}^n, \mathbb{R}^k)$ the space of jets of order p on $N_{\mathbb{R}^n}$.

The space of p -jets will be locally coordinatized by

$$x_i, u_j, (u_j)_{x_{i_1}}, (u_j)_{x_{i_1}^{j_1}, x_{i_2}^{j_2}}, \dots, (u_j)_{x_{i_1}^{j_1}, x_{i_2}^{j_2}, x_{i_3}^{j_3}, \dots, x_{i_n}^{j_n}}$$

and such that $i = 1, \dots, n$, $j = 1, \dots, k$, $j_1 + \dots + j_n \leq p$.

Let us consider a general (possibly nonlinear) system of a number q of PDEs on $J^p(\mathbb{R}^n, \mathbb{R}^k)$, with higher-order derivative of order p ,

$$\Psi^l = \Psi^l \left(x_i, u_j, (u_j)_{x_{i_1}}, (u_j)_{x_{i_1}^{j_1}, x_{i_2}^{j_2}}, \dots, (u_j)_{x_{i_1}^{j_1}, x_{i_2}^{j_2}, x_{i_3}^{j_3}, \dots, x_{i_n}^{j_n}} \right), \quad (2)$$

for all $l = 1, \dots, q$ and $i = 1, \dots, n$, $j = 1, \dots, k$, $j_1 + \dots + j_n \leq p$. The notation accords to the usual: $(u_j)_{x_{i_1}} = \partial u_j / \partial x_{i_1}$, and this definition is extensible to higher-order derivatives.

Suppose that we know a number “ n ” of conserved quantities that are

expressible in the following form

$$A_{x_i}^{(j)} \left(x_{i_1}, u_j, (u_j)_{x_{i_1}}, (u_j)_{x_{i_1}^{j_1}, x_{i_2}^{j_2}}, \dots \right) = A_{x_{i'}}^{(j')} \left(x_i, u_j, (u_j)_{x_{i_1}^{j_1}}, (u_j)_{x_{i_1}^{j_1}, x_{i_2}^{j_2}}, \dots \right),$$

$$x_i \neq x_{i'}, \quad A^{(j)} \neq A^{(j')},$$

$$1 \leq i \leq n, \quad j_1 + \dots + j_n \leq p, \quad 1 \leq j, j' \leq 2n. \quad (3)$$

where $A_{x_i}^{(j)} = \partial A^{(j)} / \partial x_i$, $A_{x_{i'}}^{(j')} = \partial A^{(j')} / \partial x_{i'}$ and $A^{(j)}, A^{(j')} \in C^\infty \mathcal{J}^p(\mathbb{R}^n, \mathbb{R}^k)$, are different.

If the number of equations in (3) is equal to the number of indepent variables, we propose a transformation for each $\{x_1, \dots, x_n\}$ to a new set $\{z_1, \dots, z_n\}$ as

$$dz_i = A^{(j)} dx_i + A^{(j')} dx_{i'}, \quad \forall 1 \leq i, i' \leq n, \quad \forall 1 \leq j, j' \leq 2n. \quad (4)$$

such that if the property of closeness is satisfied, $d^2 z_i = 0$ for all z_i , $i = 1, \dots, n$, we recover the conserved quantities given in (3).

3 Quasi-algorithmical procedure

We focus on the case in which only *one* conserved quantity equation is used. This is due to the number of physical examples in which a reciprocal transformation based on one single conserved quantity is workable. It implies that only one independent variable is transformed and it is denoted by $x_{\hat{i}}$.

We now search for a function $X(z_1, \dots, z_n)$ such that

$$x_{\hat{i}} = X(z_1, \dots, z_n) \quad (5)$$

is turned into a dependent variable. Simultaneously, we use the conserved quantity to propose the transformation

$$dz_{\hat{i}} = A^{(j)} dx_i + A^{(j')} dx_{i'},$$

$$dz_i = dx_i, \quad (6)$$

$\forall 1 \leq i \neq i' \leq n$ and for a fixed value $1 \leq \hat{i} \leq n$, where the independent variables $x_i, \forall i \neq \hat{i} = 1, \dots, n$ are untransformed but renamed as z_i .

Deriving relation (5),

$$dx_{\hat{i}} = \sum_{i=1}^n X_{z_i} dz_i, \quad X_{z_i} = \frac{\partial X}{\partial z_i} \quad (7)$$

and by isolating $dz_{\hat{i}}$ in (7), we have

$$dz_{\hat{i}} = \frac{dx_{\hat{i}}}{X_{z_{\hat{i}}}} - \sum_{i \neq \hat{i}=1}^n \frac{X_{z_i}}{X_{z_{\hat{i}}}} dx_i, \quad (8)$$

where we have used that $dz_i = dx_i$ for all $i \neq \hat{i}$ according to (6).

Here, by direct comparison of coefficients in (6) and (8), and if we identify $z_{\hat{i}}$ with z_{i_1} , we have that

$$\begin{aligned} A^{(j')} &= \frac{1}{X_{z_{\hat{i}}}}, \\ A^{(j)} &= -\frac{X_{z_i}}{X_{z_{\hat{i}}}}. \end{aligned} \quad (9)$$

Now we perform the extension of the transformation to higher-order derivatives. In the case of first-order derivatives, it is

$$\begin{aligned} u_{x_{\hat{i}}} &= \frac{\partial u}{\partial z_{\hat{i}}} \frac{\partial z_{\hat{i}}}{\partial x_{\hat{i}}} + \sum_{i \neq \hat{i}=1}^n \frac{\partial u}{\partial z_i} \frac{\partial z_i}{\partial x_{\hat{i}}} = \frac{u_{z_{\hat{i}}}}{X_{z_{\hat{i}}}}, \\ u_{x_i} &= \frac{\partial u}{\partial z_{\hat{i}}} \frac{\partial z_{\hat{i}}}{\partial x_i} + \sum_{i \neq \hat{i}=1}^n \frac{\partial u}{\partial z_i} \frac{\partial z_i}{\partial x_i} = -\frac{X_{z_i}}{X_{z_{\hat{i}}}} u_{z_{\hat{i}}} + u_{z_i}, \quad \forall i \neq \hat{i}. \end{aligned} \quad (10)$$

This process is recursively applied to achieve higher order derivatives. In this way, using expressions in (9), (10), etc., we transform a system (2) with initial variables $\{x_1, \dots, x_n\}$ into a new system written in variables $\{z_1, \dots, z_n\}$ and scalar fields $u_j(z_1, \dots, z_n)$, $\forall j = 1, \dots, k$.

From (9), we can extract expressions for

$$z_i, \quad u_j, \quad (u_j)_{z_{i_1}}, \quad (u_j)_{z_{i_1}^{j_1}, z_{i_2}^{j_2}}, \quad \dots, \quad (u_j)_{z_{i_1}^{j_1}, z_{i_2}^{j_2}, \dots, z_{i_n}^{j_n}} \quad (11)$$

for $j_1 + \dots + j_n \leq p$, if possible, given the particular form of $A^{(j)}$, $A^{(j')} \in C^\infty J^p(\mathbb{R}^n, \mathbb{R}^k)$, in each case.

Bearing in mind expression (5), the transformation of the initial system (2) will then read

$$\Psi^l = \Psi^l \left(z_i, X, X_{z_i}, X_{z_{i_1}^{j_1}, z_{i_2}^{j_2}}, \dots, X_{z_{i_1}^{j_1}, z_{i_2}^{j_2}, z_{i_3}^{j_3}, \dots, z_{i_n}^{j_n}} \right) \quad (12)$$

for all $l = 1, \dots, q$ such that $j_1 + \dots + j_n \leq p$ and $z_i = z_1, \dots, z_n$.

4 Example I: application to PDEs

4.1 The n_0 equation

The n_0 equation [12] defined on $J^4(\mathbb{R}, \mathbb{R}^3)$ has the form

$$\left(H_{x_1, x_1, x_2} + 3H_{x_2}H_{x_1} + n_0 \frac{H_{x_1 x_2}^2}{H_{x_2}} \right)_{x_1} = H_{x_2 x_3} \quad (13)$$

and has proven to be integrable in the particular cases in which $n_0 = 0, -3/4$. For these two cases, a Lax pair formulation was derived in [1, 2]. It reads

$$\begin{aligned} \phi_{x_1 x_1 x_1} - \phi_{x_3} + 3H_{x_1} \phi_{x_1} - \frac{k-5}{2} H_{x_1 x_1} \phi &= 0, \\ \phi_{x_1 x_2} + H_{x_2} \phi + \frac{k-5}{6} \frac{H_{x_1 x_2}}{H_{x_2}} \phi_{x_2} &= 0. \end{aligned} \quad (14)$$

The compatibility condition of this Lax pair ($\phi_{x_1 x_1 x_1 x_2} = \phi_{x_1 x_2 x_1 x_1}$) retrieves the n_0 equation for the integrable values

$$\begin{aligned} H_{x_1 x_1 x_2} + 3H_{x_2}H_{x_1} - \frac{k+1}{4} \frac{H_{x_1 x_2}^2}{H_{x_2}} &= 0, \\ \Omega_{x_1} &= H_{x_2 x_3} \end{aligned} \quad (15)$$

For this equation, there exists a reciprocal transformation that turns system (15) into a generalization of the Vankhnenko [38] and/or the Degasperi-Procesi equations [10] to $2+1$ dimensions.

We construct a reciprocal transformation by a change of the form

$$\begin{aligned} dx_1 &= \alpha(x, t, T) (dx - \beta(x, t, T)dt - \epsilon(x, t, T)dT), \\ x_2 &= t, \quad x_3 = T \end{aligned} \quad (16)$$

If the imposition of a closed one-form is accomplished, $d^2 x_1 = 0$, the following equations arise

$$\alpha_t + (\alpha\beta)_x = 0, \quad \alpha_T + (\alpha\epsilon)_x = 0, \quad \beta_T - \epsilon_t + \epsilon\beta_x - \epsilon_x\beta = 0 \quad (17)$$

Now we introduce $H_{x_2} = \alpha(x, t, T)^k$ that leads us to an integrability condition for the number k . That is, if $k^2 = k + 2$ is satisfied, then,

$$H_{x_1} = \frac{1}{3} \left(\frac{\Omega}{\alpha^k} - k \frac{\alpha_{xx}}{\alpha^3} + (2k-1) \left(\frac{\alpha_x}{\alpha^2} \right)^2 \right) \quad (18)$$

and using (15),

$$\Omega_x = -k\alpha^{k+1}\epsilon_x, \quad (19)$$

So,

$$\Omega_t = -\beta\Omega_x - k\Omega\beta_x + \alpha^{k-2} \left(-k\beta_{xxx} + (k-2)\beta_{xx}\frac{\alpha_x}{\alpha} + 3k\alpha^k\alpha_x \right). \quad (20)$$

The reciprocally-transformed set of equations is $\{(17), (19), (20)\}$. Nonetheless, a more convenient form arises if we introduce the following definitions

$$A_1 = \frac{k_1}{3}, \quad A_2 = \frac{2-k}{3}, \quad M = \frac{1}{\alpha^3} \quad (21)$$

Then the integrability condition turns into

$$k^2 = k + 2 \rightarrow A_1 A_2 = 0, A_1 + A_2 = 1.$$

Using this definitions, the set $\{(17), (19), (20)\}$ can be rewritten as

$$\begin{aligned} A_1 M \left(\Omega_t + \beta\Omega_x + 2\beta_x\Omega + 2\beta_{xx}\Omega + 2\frac{M_x}{M^2} \right) + \\ + A_2 (\Omega_t + \beta\Omega_x - \Omega\beta_x - M\beta_{xxx} - M_x\beta_{xx} - M_x) = 0, \\ A_1 \left(\Omega_x + 2\frac{\epsilon_x}{M} \right) + A_2 (\Omega_x - \epsilon_x) = 0, \end{aligned}$$

$$\begin{aligned} M_t = 3M\beta_x - \beta M_x, \quad M_T = 3M\epsilon_x - \epsilon M_x, \\ \beta_T - \epsilon_t + \epsilon\beta_x - \epsilon\beta_x = 0. \end{aligned} \quad (22)$$

The transformation can also be applied to the spectral problem and after some direct calculations, we have

$$\psi_{xt} = A_1 \left(-\beta\psi_{xx} + \left(\beta_{xx} - \frac{1}{M} \right) \psi \right) + A_2 (-\beta\psi_{xx} - 2\beta_x\psi_x - (1 + \beta_{xx})),$$

$$\begin{aligned} \psi_T = A_1 (M\psi_{xxx} + (M\omega - \epsilon)\psi_x) + \\ + A_2 (M\psi_{xxx} + 2M_x\psi_{xx} + (M_{xx} + \Omega - \epsilon)\psi_x) \end{aligned} \quad (23)$$

where we have set

$$\phi(x_1, x_2, x_3) = \alpha^{\frac{2k-1}{3}} \psi(x, t, T)$$

4.1.1 Reduction

We reduce set of equations in (22) by setting $\epsilon = 0, \Omega = a_0$. The system reduces to

$$\begin{aligned} 2MA_1 \left(\beta_{xx} + a_0\beta - \frac{1}{M} \right)_x - A_2 (M\beta_{xx} + a_0\beta + M)_x &= 0, \\ M_t &= 3M\beta_x - \beta M_x. \end{aligned} \quad (24)$$

The reduction of the Lax pair can be obtained by setting $\psi_T = \lambda\psi$. In this case, the reduced spectral problem is

$$\begin{aligned} A_1 \left(\psi_{xxx} + a_0\psi_x - \frac{\lambda}{M}\psi \right) + \\ + A_2 \left(\psi_{xxx} + 2\frac{M_x}{M}\psi_{xx} + \frac{1}{M}(M_{xx} + a_0)\psi_x - \frac{\lambda}{M}\psi \right) &= 0 \end{aligned} \quad (25)$$

and

$$\begin{aligned} A_1 (\lambda\psi_t + \psi_{xx} + \lambda\beta\psi_x + (a_0 - \lambda\beta_x)\psi) + \\ + A_2 (\lambda\psi_t + M\psi_{xx} + (\lambda\beta + M_x)\psi_x + (a_0 + \lambda\beta_x)\psi) &= 0 \end{aligned} \quad (26)$$

Degasperis-Procesi equation

For the case $A_1 = 0$ and $A_2 = 1$, we can integrate the reduction as

$$\begin{aligned} \beta_{xx} + a_0\beta &= \frac{1}{M} + q_0, \\ (\beta_{xx} + a_0\beta)_t + \beta\beta_{xxx} + 3\beta_x\beta_{xx} + 4a_0\beta\beta_x - 3q_0\beta_x &= 0. \end{aligned} \quad (27)$$

For $q_0 = 0$ and $a_0 = -1$, we retrieve the well-known Degasperis-Procesi equation. The reduced Lax pair is in accordance with

$$\begin{aligned} \psi_{xxx} - \psi_x - \lambda(\beta_{xx} - \beta)\psi &= 0, \\ \lambda\psi_t + \psi_{xx} + \lambda\beta\psi_x - (1 + \lambda\beta_x)\psi &= 0 \end{aligned} \quad (28)$$

which is equivalent to the Degasperi-Procesi Lax pair [10].

Vakhnenko equation

For the case $A_1 = 0, A_2 = 1$ and $a_0 = 0$, we integrate the equations as

$$\beta_{xx} + 1 = \frac{q_0}{M},$$

$$((\beta_t + \beta\beta_x)_x + 3\beta)_x = 0 \quad (29)$$

which is the derivative of the Vakhnenko equation [38], whose Lax pair is

$$\begin{aligned} \psi_{xxx} + \frac{M_x}{M}\psi_{xx} + \frac{M_{xx}}{M}\psi_x - \frac{\lambda}{M}\psi &= 0, \\ \lambda\psi_t + M\psi_{xx} + (\lambda\beta + M_x)\psi_x + \lambda\beta_x\psi &= 0. \end{aligned} \quad (30)$$

This was the first time that a Lax pair for the Vakhnenko equation had been identified by one of the present authors [12].

5 Example II: application to hierarchies of PDEs

The forthcoming examples will show the application of this procedure to the particular case in which the number of initial independent variables is 3, with the identification $x_1 = x, x_2 = y, x_3 = t$. In order to make things clearer, some slight changes in the notation of the theory or the dimension of $N_{\mathbb{R}^n}$ shall be altered to fit our concrete examples.

5.1 The Camassa–Holm hierarchy

The Camassa–Holm hierarchy in $(2+1)$ dimensions (henceforth CHH(2+1)) can be written in a compact form as

$$U_T = R^{-n}U_Y, \quad (31)$$

where R is the recursion operator defined by the composition of two operators K and J

$$R = JK^{-1}, \quad K = \partial_{XXX} - \partial_X, \quad J = -\frac{1}{2}(\partial_X U + U\partial_X). \quad (32)$$

The n component of this hierarchy can also be rewritten as a set of PDEs by introducing n dependent fields $\Omega_{[i]}, (i = 1, \dots, n)$ in the following way

$$\begin{aligned} U_Y &= J\Omega_{[1]}, \\ J\Omega_{[i+1]} &= K\Omega_{[i]}, \quad i = 1, \dots, n-1, \\ U_T &= K\Omega_{[n]}, \end{aligned} \quad (33)$$

and by introducing two new fields, P and Δ , related with U as

$$U = P^2, \quad P_T = \Delta_X, \quad (34)$$

we can write the hierarchy in the form of the following set of equations

$$\begin{aligned} P_Y &= -\frac{1}{2} (P\Omega_{[1]})_X, \\ (\Omega_{[i]})_{XXX} - (\Omega_{[i]})_X &= -P (P\Omega_{[i+1]})_X, \quad i = 1, \dots, n-1, \\ P_T &= \frac{(\Omega_{[n]})_{XXX} - (\Omega_{[n]})_X}{2P} = \Delta_X. \end{aligned} \quad (35)$$

According to (3), the conservative form of the first two equations

$$\begin{aligned} A^{(1)} &= P, x_1 = Y, \quad A^{(2)} = -\frac{1}{2} (P\Omega_{[1]})_X, x_2 = X \quad \text{or} \\ A^{(1)} &= P, x_1 = T, \quad A^{(2)} = \Delta, x_2 = X. \end{aligned} \quad (36)$$

allows us to define the exact derivative

$$dz_0 = P dX - \frac{1}{2} P\Omega_{[1]} dY + \Delta dT. \quad (37)$$

A reciprocal transformation can be introduced by considering the former independent variable X as a field depending on z_0 , $z_1 = Y$ and $z_{n+1} = T$, such that $d^2X = 0$. From (37) we have

$$\begin{aligned} dX &= \frac{1}{P} dz_0 + \frac{\Omega_{[1]}}{2} dz_1 - \frac{\Delta}{P} dz_{n+1}, \\ Y &= z_1, \\ T &= z_{n+1}, \end{aligned} \quad (38)$$

If we consider the new field $X = X(z_0, z_1, \dots, z_{n+1})$, by direct comparison

$$\begin{aligned} X_{z_0} &= \frac{1}{P}, \\ X_{z_1} &= \frac{\Omega_{[1]}}{2}, \\ X_{z_{n+1}} &= -\frac{\Delta}{P}, \end{aligned} \quad (39)$$

where $X_{z_i} = \frac{\partial X}{\partial z_i}$. We can now extend the transformation by introducing a new independent variable z_i for each field $\Omega_{[i]}$ by generalizing (39) as

$$X_{z_i} = \frac{\Omega_{[i]}}{2}, \quad i = 1, \dots, n. \quad (40)$$

Each of the former dependent fields $\Omega_{[i]}$, ($i = 1, \dots, n$) allows us to define a new dependent variable z_i through definition (40). It requires some

calculation (see [19] for details) but it can be proven that the reciprocal transformation (38)-(40) transforms (35) to the following set of n PDEs on $J^4(\mathbb{R}, \mathbb{R}^{n+2})$

$$-\left(\frac{X_{z_{i+1}}}{X_{z_0}}\right)_{z_0} = \left[\left(\frac{X_{z_0, z_0}}{X_{z_0}} + X_{z_0}\right)_{z_0} - \frac{1}{2} \left(\frac{X_{z_0, z_0}}{X_{z_0}} + X_{z_0}\right)^2 \right]_{z_i}, \quad (41)$$

with $i = 1, \dots, n$. Note that each equation depends on only three variables z_0, z_i, z_{i+1} . This result generalizes the one found in [26] for the first component of the hierarchy. The conservative form of (41) allows us to define a field $M(z_0, z_1, \dots, z_{n+1})$ such that

$$\begin{aligned} M_{z_i} &= -\frac{1}{4} \left(\frac{X_{z_{i+1}}}{X_{z_0}}\right), \quad i = 1, \dots, n, \\ M_{z_0} &= \frac{1}{4} \left[\left(\frac{X_{z_0, z_0}}{X_{z_0}} + X_{z_0}\right)_{z_0} - \frac{1}{2} \left(\frac{X_{z_0, z_0}}{X_{z_0}} + X_{z_0}\right)^2 \right]. \end{aligned} \quad (42)$$

It is easy to prove that each M_i should satisfy the following Calogero-Bogoyanlevskii-Schiff equation (the CBS equation) [5, 7] on $J^4(\mathbb{R}, \mathbb{R}^{n+2})$

$$M_{z_0, z_{i+1}} + M_{z_0, z_0, z_i} + 4M_{z_i} M_{z_0, z_0} + 8M_{z_0} M_{z_0, z_i} = 0, \quad (43)$$

with $i = 1, \dots, n$.

The CBS equation has the Painlevé property and the SMM can be successfully used to derive its Lax pair [17]. In [19] it was proven that the Lax pair of CBS yields the following spectral problem for the CHH(2+1) hierarchy (33)

$$\begin{aligned} \Phi_{XX} + \frac{1}{4}(\lambda U - 1)\Phi &= 0, \\ \Phi_T - \lambda^n \Phi_Y - \frac{\lambda}{2} C \Phi_X + \frac{\lambda}{4} C_X \Phi &= 0 \end{aligned} \quad (44)$$

where

$$C = \sum_{i=1}^n \lambda^{n-i} \Omega_{[i]}$$

and $\lambda(Y, T)$ is a non-isospectral parameter that satisfies

$$\lambda_X = 0, \quad \lambda_T - \lambda^n \lambda_Y = 0. \quad (45)$$

Consequently the problems that we meet when we try to apply the Painlevé test to CHH(2+1) [25] can be solved owing to the existence of a reciprocal transformation that transforms the CHH(2+1) hierarchy to n copies of the CBS equation, for which the Painlevé methods are applicable.

5.2 The modified Camassa–Holm hierarchy

In [13], the modified Camassa–Holm hierarchy in $(2+1)$ dimensions (henceforth mCHH $(2+1)$), was introduced

$$u_t = r^{-n} u_y, \quad (46)$$

where r is the recursion operator, defined by two operators j and k and generalizes the hierarchy introduced by Qiao [28]

$$r = jk^{-1}, \quad k = \partial_{xxx} - \partial_x, \quad j = -\partial_x u (\partial_x)^{-1} u \partial_x. \quad (47)$$

If we introduce $2n$ auxiliary fields $v_{[i]}$, $\omega_{[i]}$ defined through

$$\begin{aligned} u_y &= jv_{[1]}, \\ jv_{[i+1]} &= kv_{[i]}, \\ (\omega_{[i]})_x &= u(v_{[i]})_x, \\ u_t &= kv_{[n]}, \end{aligned} \quad i = 1, \dots, n-1, \quad (48)$$

the hierarchy can be written as the system

$$\begin{aligned} u_y &= -(u\omega_{[1]})_x, \\ (v_{[i]})_{xxx} - (v_{[i]})_x &= -(u\omega_{[i+1]})_x, \\ u_t &= (v_{[n]})_{xxx} - (v_{[n]})_x = \delta_x, \end{aligned} \quad i = 1, \dots, n-1, \quad (49)$$

According to the theory in (3),

$$\begin{aligned} A^{(1)} = u, x_1 = y, \quad A^{(2)} = -u\omega_{[1]}, x_2 = x \quad \text{or} \\ A^{(1)} = u, x_1 = t, \quad A^{(2)} = \delta, x_2 = x, \end{aligned} \quad (50)$$

which allows to define the exact derivative

$$dz_0 = u dx - u\omega_{[1]} dy + \delta dt \quad (51)$$

and $z_1 = y, z_{n+1} = t$. We can define a reciprocal transformation such that the former independent variable x is a new field $x = x(z_0, z_1, \dots, z_{n+1})$ depending on $n+2$ variables in the form

$$\begin{aligned} x_{z_0} &= \frac{1}{u}, \\ x_{z_1} &= \omega_{[1]}, \\ x_{z_{n+1}} &= -\frac{\delta}{u}. \end{aligned} \quad (52)$$

If we introduce the auxiliary variables for the auxiliary fields, $x_{z_i} = \omega_{[i]}$ for $i = 2, \dots, n$, the transformation of the equations in (49) yields a system of equations on $J^4(\mathbb{R}^{n+2}, \mathbb{R})$. Note that each equation depends on three variables: z_0, z_i, z_{i+1} .

$$\left(\frac{x_{z_{i+1}}}{x_{z_0}} + \frac{x_{z_i, z_0, z_0}}{x_{z_0}} \right)_{z_0} = \left(\frac{x_{z_0}^2}{2} \right)_{z_i}, \quad i = 1, \dots, n. \quad (53)$$

The conservative form of (53) allows us to define a field $m = m(z_0, z_1, \dots, z_{n+1})$ such that on $J^3(\mathbb{R}^2, \mathbb{R}^{n+2})$ we have

$$m_{z_0} = \frac{x_{z_0}^2}{2}, \quad m_{z_i} = \frac{x_{z_{i+1}}}{x_{z_0}} + \frac{x_{z_i, z_0, z_0}}{x_{z_0}}, \quad i = 1, \dots, n. \quad (54)$$

Equation (53) has been extensively studied from the point of view of Painlevé analysis [17] and it can be considered as the modified version of the CBS equation (mCBS) (43). Actually, in [17] it was proven that the *Miura transformation* that relates (43) and (54) is

$$4M = x_{z_0} - m. \quad (55)$$

A non-isospectral Lax pair was obtained for (54) in [17]. By inverting this Lax pair through the reciprocal transformation (52) the following spectral problem was obtained for mCHH(2+1). This Lax pair [17] reads

$$\begin{aligned} \begin{pmatrix} \phi \\ \hat{\phi} \end{pmatrix}_x &= \frac{1}{2} \begin{bmatrix} -1 & I\sqrt{\lambda}u \\ I\sqrt{\lambda}u & 1 \end{bmatrix} \begin{pmatrix} \phi \\ \hat{\phi} \end{pmatrix}, \\ \begin{pmatrix} \phi \\ \hat{\phi} \end{pmatrix}_t &= \lambda^n \begin{pmatrix} \phi \\ \hat{\phi} \end{pmatrix}_y + \lambda a \begin{pmatrix} \phi \\ \hat{\phi} \end{pmatrix}_x + I \frac{\sqrt{\lambda}}{2} \begin{bmatrix} 0 & b_{xx} - b_x \\ b_{xx} + b_x & 0 \end{bmatrix} \begin{pmatrix} \phi \\ \hat{\phi} \end{pmatrix}, \end{aligned} \quad (56)$$

$$(57)$$

where

$$a = \sum_{i=1}^n \lambda^{n-i} \omega_{[i]}, \quad b = \sum_{i=1}^n \lambda^{n-i} v_{[i]}, \quad I = \sqrt{-1} \quad i = 1, \dots, n, \quad (58)$$

and $\lambda(y, t)$ is a non-isospectral parameter that satisfies

$$\lambda_x = 0, \quad \lambda_t - \lambda^n \lambda_y = 0. \quad (59)$$

Although the Painlevé test cannot be applied to mCHH(2+1), reciprocal transformations are a tool that can be used to write the hierarchy as a set of mCBS equations to which the Painlevé analysis (the SMM in particular) can be successfully applied.

6 Conclusions

We have first proposed reciprocal transformations for a single PDE. We have found two interesting reductions for the transformed equation of our proposed model. One corresponds to the Vakhnenko equation, the other is a Degasperis–Procesi type equation. In this way, we can say that our proposed equation is a generalization to a higher dimension of the mentioned equations.

Secondly, we have proposed reciprocal transformations for complete hierarchies of PDEs. In particular, we have contemplated the CHH(2 + 1) and the mCHH(2 + 1). We have presented both hierarchies and have discussed some general properties for a scalar field $U(X, Y, T)$ and $u(x, y, t)$ respectively. We have constructed reciprocal transformations that connect both hierarchies with the CBS and mCBS equations, respectively. The Lax pair for both of the hierarchies can be retrieved through those of the CBS and mCBS, correspondingly. If we consider the Lax pair of the CBS and mCBS and undo the reciprocal transformation, we achieve the Lax pair of the hierarchies.

As illustrated through the examples, we can say that reciprocal transformations help us reduce the number of available nonlinear equations in the literature, as two seemingly different equations can be turned from one into another by reciprocal transformation, which is the case of the CHH(2 + 1), mCHH(2 + 1) with the CBS and mCBS, correspondingly. A question for future work is whether there exists a canonical description for differential equations. Intuitively, we expect that if two equations are essentially the same, although apparently different in disguised versions, they must share the same singular manifold equations. But this is just an initial guess worth of further research in the future.

Also, reciprocal transformations have proven their utility in the derivation of Lax pairs. As we know, obtaining Lax pairs is a nontrivial subject. The common way is to impose ad hoc forms for such linear problems and make them fit according to the compatibility condition. Accounting for reciprocal transformations, we do not face the problem of imposing ad hoc Ansätze. One initial equation whose Lax pair is unknown, can be interpreted as the reciprocally-transformed equation whose Lax pair is acknowledged. In this way, undoing the transformation on the latter Lax pair, we achieve the Lax pair of the former. This is precisely the procedure followed along our examples.

As a last property, to mention that reciprocal transformations permit us to (sometimes) obtain equations integrable in the Painlevé sense if they did

not have this property before the change. This is due to the noninvariability of the Painlevé test under changes of variables.

Some possible future research on this topic would consist of understanding whether the singular manifold equations can constitute a canonical representation of a partial differential equation and designing techniques to derive Lax pairs in a more unified way. Also, the trial of composition of reciprocal transformations with transformations of other nature, can lead to more unexpected but desirable results.

7 References

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